

# Minimal Irreversible Quantum Mechanics: Pure States Formalism

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## Abstract

It is demonstrated that, making minimal changes in ordinary quantum mechanics, a reasonable irreversible quantum mechanics can be obtained. This theory has a more general spectral decompositions, with eigenvectors corresponding to unstable states that vanish when  $t \rightarrow \infty$ . These "Gamov vectors" have zero norm, in such a way that the norm and the energy of the physical states remain constant. The evolution operator has no inverse, showing that we are really dealing with a time-asymmetric theory. Using Friedrichs model reasonable physical results are obtained, e. g. : the remaining of an unstable decaying state reappears, in the continuous spectrum of the model, with its primitive energy.

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# 1 Introduction.

Usually rigorous quantum mechanics must be formulated in a Gel'fand triplet [1]:

$$\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^\times, \quad (1)$$

where:

$\mathcal{S}$  is the space of "regular states" or test functions space, that correspond to Schwarz class wave functions, that are considered as the physical states.

$\mathcal{H}$  is the space of "states", or Hilbert space, introduced to extend the notion of probability to a larger space. These states are square integrable wave functions, e. g. Schwarz functions where a finite set of points is removed from the curve representing the function. As it is not clear for us what is the physical meaning of this kind vectors, in this paper we will consider that only  $\mathcal{S}$  contains the physical states.

$\mathcal{S}^\times$  is the space of "generalized states", or rigged Hilbert space, namely the space of linear functionals over  $\mathcal{S}$ , that essentially are used to find the spectral expansion of the regular states.

Let  $K$  be the Wigner or time-inversion operator. In the usual time-symmetric or reversible quantum mechanics the evolution Hamiltonian  $H$  is time symmetric, i. e.:

$$KHK^\dagger = H. \quad (2)$$

In fact, if it were time-asymmetric the theory would be trivially time asymmetric, and we know that such a trivial theory do not coincide with physical reality. In wave function representation  $K$  coincide with complex conjugation, so it is defined over  $\mathcal{S}$  by:

$$K\varphi(x) = \varphi^*(x), \quad (3)$$

$$K : \mathcal{S} \rightarrow \mathcal{S}. \quad (4)$$

So  $\mathcal{S}$  space is also time-symmetric.

But the real universe and macroscopic objects have clearly time-asymmetric evolutions, so we must explain how these time-asymmetry appears if the quantum mechanical basic laws of the universe (embodied in  $\mathcal{H}$ ) are time-symmetric. The usual and successful explanation is based in coarse grainig: macroscopic objects have a huge number of dynamical variables and we can

measure and control only a small number of them, the so call relevant variables. If we neglect the rest of the variables, the irrelevant ones, we obtain time-asymmetric evolution equations [2].

Nevertheless in this paper (according to the line of thought pioneered in references [4][5][6]) we want to follow a different way, because we believe that the development of an alternative theory will enhance our knowledge about time-asymmetry. Thus we want to sketch an irreversible quantum theory, which explains time-asymmetry from the basic microscopic level directly. In this way we will have two (probably equivalent) theories to compare.

Obviously we want to obtain our new theory making minimal changes to the well established and usual quantum mechanics. If we change eqs. 2 or 3 it is almost sure to find experimental problems. So the minimal modification is to change eq. 4 defining a new test functions space  $\phi_- \subset \mathcal{S}$  such that:

$$K : \phi_- \rightarrow \phi_+ \neq \phi_- . \quad (5)$$

In this way  $K$  is not even defined over the space of regular states  $\phi_-$  and time-asymmetry naturally appears.

We shall demonstrate that an irreversible quantum theory based in a Gel'fand triplet:

$$\phi_- \subset \mathcal{H} \subset \phi_-^\times, \quad (6)$$

is feasible and it yield reasonable physical results, as the decaying of unstable states, if test function space  $\phi_-$  is properly chosen. We shall show that, what it is done in the quoted papers [4][5][6], is essentially our minimal modification of the ordinary reversible quantum theory. But with this new approach we gain a more clear comprehension of the extension from the reversible quantum theory to the irreversible one, described in these papers.

The paper is organized as follows:

-In section 2 we review the analytic extension method used to obtain new spectral decompositions: the main new tool of the formalism. Complex eigenvalues appear in this expansion corresponding to unstable states or Gamov vectors.

-In section 3 it is proved that the norm of these Gamov vectors vanish, showing that they are not physical states but only "ghosts", that we can use to make simpler our computations. This fact is essential to preserve the norm and the energy of the physical states.

-In section 4 we prove that, in our theory, the time evolution operator has no inverse. This fact shows that this theory is really time asymmetric. We can then discuss the origin of the arrow of time.

-In section 5 the Friedrichs model is introduced and some reasonable physical results are obtained.

In this paper we only deal with pure states. The study of mixed states is in progress and will be the subject of another paper [7].

## 2 Generalized spectral decomposition by analytic extensions.

### 2.1 Usual spectral decomposition of $H_0$ .

Let us consider first  $\mathcal{S}$  as the space of regular wave vectors in energy representation.

The internal product of two wave vectors  $\varphi$  and  $\psi$ , represented by  $\varphi(\omega)$  and  $\psi(\omega)$ , both belonging to  $\mathcal{S}$ , is given by

$$\langle \varphi | \psi \rangle = \int_0^\infty d\omega \varphi^*(\omega) \psi(\omega) \quad (7)$$

Let us suppose that the free Hamiltonian operator  $H_0$  satisfy

$$\langle \varphi | H_0 \psi \rangle = \langle H_0 \varphi | \psi \rangle = \int_0^\infty d\omega \varphi^*(\omega) \omega \psi(\omega) \quad (8)$$

If we now introduce the linear (antilinear) functionals  $\langle \omega |$  ( $|\omega\rangle$ ) on  $\mathcal{S}$ , defined by

$$\langle \omega | \psi \rangle \equiv \psi(\omega), \quad \langle \varphi | \omega \rangle \equiv \varphi^*(\omega), \quad (9)$$

expressions 2.1 and 2.2 can be written as

$$\langle \varphi | \psi \rangle = \int_0^\infty d\omega \langle \varphi | \omega \rangle \langle \omega | \psi \rangle, \quad (10)$$

$$\langle \varphi | H_0 \psi \rangle = \int_0^\infty d\omega \langle \varphi | \omega \rangle \omega \langle \omega | \psi \rangle \quad (11)$$

If we omit the 'bra'  $\langle \varphi |$  ('ket'  $|\psi\rangle$ ) in 2.4 we obtain the following formal expression for  $|\psi\rangle$  ( $\langle \varphi |$ ) in terms of the functionals  $|\omega\rangle$  ( $\langle \omega |$ ):

$$|\psi\rangle = \int_0^\infty d\omega |\omega\rangle \langle \omega | \psi \rangle, \quad \langle \varphi | = \int_0^\infty d\omega \langle \varphi | \omega \rangle \langle \omega | \quad (12)$$

Equations 2.6 yield equation 2.4 for the product  $\langle\varphi|\psi\rangle$ , if we impose on the functionals defined in eq. 2.3 the generalized orthogonality condition

$$\langle\omega|\omega'\rangle = \delta(\omega - \omega'). \quad (13)$$

From 2.4 and 2.5 we can also obtain the formal expressions

$$I = \int_0^\infty d\omega |\omega\rangle\langle\omega| \quad (14)$$

$$H_0 = \int_0^\infty d\omega |\omega\rangle\omega\langle\omega| \quad (15)$$

Equation 2.9 is the usual spectral decomposition of  $H_0$ , satisfying

$$\begin{aligned} \langle\omega|H_0|\psi\rangle &= \omega\langle\omega|\psi\rangle & \psi \in \mathcal{S} \\ \langle\varphi|H_0|\omega\rangle &\equiv \langle H_0\varphi|\omega\rangle = \omega\langle\varphi|\omega\rangle & \varphi \in \mathcal{S} \end{aligned} \quad (16)$$

## 2.2 Complex spectral decomposition of $H_0$ .

In order to obtain more general spectral expansions than the usual ones we would like to promote the real variable  $\omega$  to a complex variable  $z$  and to change the integral over  $\mathcal{R}^+$  in equation 2.4 by an integral over a curve  $\Gamma$  of the complex plane, as in figure 1. This change can be done if  $|\psi\rangle \in \Psi \subset \mathcal{S} \subset \mathcal{H}$  and  $|\varphi\rangle \in \Phi \subset \mathcal{S} \subset \mathcal{H}$ , where  $\Psi$  and  $\Phi$  are subspaces of  $\mathcal{S} \subset \mathcal{H}$ , for which the analytic extensions  $\psi(z)$  of  $\psi(\omega)$  and  $\varphi^\#(z)$  of  $\varphi^*(\omega)$  are well defined at least in the shadowed region of fig. 1. We have introduced the notation  $\varphi^\#(z) = [\varphi(z^*)]^*$ . Thus, we can write

$$\langle\varphi|\psi\rangle = \int_\Gamma dz \varphi^\#(z) \psi(z), \quad \varphi \in \Phi, \quad \psi \in \Psi \quad (17)$$

We define the linear (antilinear) functionals  $\langle z|$  ( $|z\rangle$ ) on  $\Psi$  ( $\Phi$ ), by

$$\begin{aligned} \langle z|\psi\rangle &\equiv \psi(z), & \psi \in \Psi \\ \langle\varphi|z\rangle &\equiv \varphi^\#(z), & \varphi \in \Phi \end{aligned} \quad (18)$$

and therefore

$$\langle\varphi|\psi\rangle = \int_\Gamma dz \langle\varphi|z\rangle \langle z|\psi\rangle \quad (19)$$

$$\langle\varphi|H_0\psi\rangle = \int_\Gamma dz \langle\varphi|z\rangle z \langle z|\psi\rangle \quad (20)$$

We can also write the formal expressions

$$\begin{aligned} |\psi\rangle &= \int_{\Gamma} dz |z\rangle \langle z|\psi\rangle \\ \langle\varphi| &= \int_{\Gamma} dz \langle\varphi|z\rangle \langle z| \end{aligned} \quad (21)$$

From 2.15 we obtain 2.13 if we impose the generalized orthogonality condition

$$\langle z|z'\rangle = \delta_{\Gamma}(z - z'), \quad (22)$$

where  $\delta_{\Gamma}$  is defined by the equation

$$\int_{\Gamma} dz g(z) \delta_{\Gamma}(z - z) = g(z),$$

being  $g(z)$  an adequate test function defined on  $\Gamma$ .

From equations 2.13 and 2.14 we obtain the formal expressions

$$I = \int_{\Gamma} dz |z\rangle \langle z| \quad (23)$$

$$H_0 = \int_{\Gamma} dz |z\rangle z \langle z| \quad (24)$$

Equation 2.18 is the new spectral decomposition for  $H_0$ .

By the analytic extension of equations 2.10 we obtain

$$\begin{aligned} \langle z|H_0\psi\rangle &= z\langle z|\psi\rangle & \psi \in \Psi \\ \langle\varphi|H_0z\rangle &\equiv \langle H_0\varphi|z\rangle = z\langle\varphi|z\rangle & \varphi \in \Phi \end{aligned} \quad (25)$$

Thus,  $|z\rangle$  and  $\langle z|$  are generalized right and left eigenvectors of  $H_0$  ( $z$  belong to the shadowed region of figure 1).<sup>1</sup>. So formally:

$$\langle z|H_0 = z\langle z| \quad H_0|z\rangle = z|z\rangle$$

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<sup>1</sup>In the usual spectral decomposition of  $H_0$  we have, according to 2.3:

$$\langle\varphi|\omega\rangle = [\varphi(\omega)]^* = [\langle\omega|\varphi\rangle]^*,$$

and therefore the adjoint generalized state of  $|\omega\rangle$  is  $\langle\omega|$ .

This is not the case for the complex spectral decomposition. According to our definition 2.12

$$\langle\varphi|z\rangle = \varphi^{\#}(z) \equiv [\varphi(z^*)]^* = [\langle z^*|\varphi\rangle]^*,$$

and the adjoint generalized state of  $|z\rangle$  is, in this case,  $\langle z^*|$ .

### 2.3 Usual spectral decomposition of H.

Up to now, we have obtained analytical continuations starting from the complete set  $\{|\omega\rangle\}$  of generalized eigenvectors of the unperturbed Hamiltonian  $H_0$ . We can, as well, start from the generalized eigenvectors  $\{|\omega_\pm\rangle\}$  of the total Hamiltonian  $H = H_0 + V$ , given by the solutions of the Lipmann-Schwinger equation:

$$\begin{aligned} |\omega_\pm\rangle &= |\omega\rangle + \frac{1}{\omega \pm io - H_0} V |\omega_\pm\rangle, \\ \langle\omega_\pm| &= \langle\omega| + \langle\omega_\pm| V \frac{1}{\omega \mp io - H_0}, \end{aligned} \quad (26)$$

satisfying:

$$\langle\omega_\pm|\omega'_\pm\rangle = \delta(\omega - \omega'), \quad |\omega_+\rangle = S(\omega)|\omega_-\rangle, \quad \langle\omega_+| = S^*(\omega)\langle\omega_-|$$

In the last expressions,  $S(\omega)$  is the trace of the S-matrix, satisfying:

$$S(\omega) S^*(\omega) = 1$$

If the Lipmann-Schwinger solutions form a complete orthonormal set we have, in weak sense (as in eqs. 2.8 and 2.9 ):

$$I = \int_0^\infty d\omega |\omega_\pm\rangle \langle\omega_\pm|, \quad H = \int_0^\infty d\omega |\omega_\pm\rangle \omega \langle\omega_\pm|.$$

These are the usual spectral expansions.

### 2.4 Complex spectral decomposition of H.

Let us now consider vectors  $|\psi\rangle \in \Psi \subset \mathcal{S} \subset \mathcal{H}$  and  $|\varphi\rangle \in \Phi \subset \mathcal{S} \subset \mathcal{H}$ , where  $\Psi$  and  $\Phi$  are defined in such a way that  $\langle\omega_\pm|\psi\rangle$  and  $\langle\varphi|\omega_\pm\rangle$  have analytic extensions  $\langle z_\pm|\psi\rangle$  and  $\langle\varphi|z_\pm\rangle$  at least in the shadowed area of fig. 1 where the analytic extension  $S(z)$  of  $S(\omega)$  has no poles. Repeating the formalism of section 2.2, the functionals  $|z_\pm\rangle$  and  $\langle z_\pm|$  can be used in the expansions of  $|\psi\rangle$  and  $\langle\varphi|$ , that is:

$$|\psi\rangle = \int_\Gamma dz |z_\pm\rangle \langle z_\pm|\psi\rangle, \quad \langle\varphi| = \int_\Gamma dz \langle\varphi|z_\pm\rangle \langle z_\pm|, \quad (27)$$

from which we obtain:

$$\langle \varphi | \psi \rangle = \int_{\Gamma} dz \langle \varphi | z_{\pm} \rangle \langle z_{\pm} | \psi \rangle \quad (28)$$

provided that:

$$\langle z_{\pm} | z'_{\pm} \rangle = \delta_{\Gamma}(z - z'). \quad (29)$$

We also have the generalized spectral decompositions:

$$I = \int_{\Gamma} dz |z_{\pm}\rangle \langle z_{\pm}|, \quad H = \int_{\Gamma} dz |z_{\pm}\rangle z \langle z_{\pm}|, \quad (30)$$

and the functionals  $|z_{\pm}\rangle$  and  $\langle z_{\pm}|$  are generalized right and left eigenvectors of the total Hamiltonian  $H$ :

$$\begin{aligned} \langle z_{\pm} | H \psi \rangle &= z \langle z_{\pm} | \psi \rangle & \psi \in \Psi \\ \langle \varphi | H | z_{\pm} \rangle &\equiv \langle H \varphi | z_{\pm} \rangle = z \langle \varphi | z_{\pm} \rangle & \varphi \in \Phi \end{aligned} \quad (31)$$

We also have the relations:

$$|z_{+}\rangle = S(z)|z_{-}\rangle, \quad \langle z_{+}| = S^{\#}(z)\langle z_{-}|, \quad (32)$$

being  $S(z)$  and  $S^{\#}(z)$  the meromorphic extensions of  $S(\omega)$  and  $S^{*}(\omega)$ .

The previous equations are valid in a weak or functional sense and only in the shadowed area of fig 1. In general the domains of analyticity of  $\langle \varphi | z_{\pm} \rangle$  and  $\langle z_{\pm} | \psi \rangle$  cannot be bigger than the domains in which  $\langle \varphi | z \rangle$  and  $\langle z | \psi \rangle$  are analytic, as can be seen from eq. 2.20, since usually the second factors of the analytic extensions of the right hand sides have poles. The domains of analyticity of  $\langle \varphi | z_{\pm} \rangle$  and  $\langle z_{\pm} | \psi \rangle$  are a consequence of the presence of these poles which are also poles of the S-matrix. For simplicity we shall assume that  $S(z)$  has a single pole at  $z_0 \in \mathcal{C}^{-}$  (the lower complex half plane) and therefore  $S^{\#}(z)$  has a single pole at  $z_0^{*} \in \mathcal{C}^{+}$  (the upper complex half plane).

Taking into account the presence of these poles different spectral decompositions of  $I$  and  $H$  can be obtained. If we chose a curve  $\Gamma_d$ , as in fig. 2<sub>a</sub>, the integral is equal to the sum of the corresponding integrals over the curves  $C_d$  and  $C'_d$  of fig. 2<sub>b</sub> and we obtain a new spectral expansion for  $I$  (see [5] for more details):

$$\begin{aligned} I = \int_{\Gamma_d} dz |z_{+}\rangle \langle z_{+}| &= \int_{C_d} dz |z_{+}\rangle \langle z_{+}| + \oint_{\tilde{C}'_d} dz S(z) |z_{-}\rangle \langle z_{+}| = \\ &= \int_{C_d} dz |f_z\rangle \langle f_z| + |f_0\rangle \langle f_0|, \end{aligned} \quad (33)$$



where

$$\begin{aligned} |f_z\rangle &= |z_+\rangle, & \langle \tilde{f}_z | &= \langle z_+|, \\ |f_0\rangle &= [-2\pi i(Res S)_{z_0}]^{\frac{1}{2}}|z_{0-}\rangle, & \langle \tilde{f}_0 | &= [-2\pi i(Res S)_{z_0}]^{\frac{1}{2}}\langle z_{0+}|. \end{aligned} \quad (34)$$

We emphasize that 2.27 is a formal expression which acquires meaning when it is 'sandwiched' between  $\langle \varphi |$  and  $|\psi\rangle$  ( $\varphi \in \Phi$  and  $\psi \in \Psi$ ).

Orthogonality relations between the vectors defined in eqs. 2.28 can be obtained using eqs.2.27 and 2.23. Consider, for example:

$$\begin{aligned} |f_0\rangle\langle \tilde{f}_0 | f_0\rangle\langle \tilde{f}_0 | &= \oint_{C'_d} dz \oint_{C'_d} dz' |z_+\rangle\langle z_+|z'_+\rangle\langle z'_+| = \\ &= \oint_{C'_d} dz \oint_{C'_d} dz' \delta_{C'_d}(z - z') |z_+\rangle\langle z'_+| = \\ &= \oint_{C'_d} dz |z_+\rangle\langle z_+| = |f_0\rangle\langle \tilde{f}_0 | \end{aligned}$$

From the first and last term of this equation we can deduce that:

$$\langle \tilde{f}_0 | f_0\rangle = 1, \quad (35)$$

Using analogous arguments we can obtain:

$$\begin{aligned} \langle \tilde{f}_0 | f_z\rangle &= \langle \tilde{f}_z | f_0\rangle = 0, \\ \langle \tilde{f}_z | f_{z'}\rangle &= \delta_{C_d}(z - z'). \end{aligned} \quad (36)$$

In terms of the functionals defined in eq. 2.28 we obtain a new spectral expansion for the Hamiltonian:

$$H = z_0 |f_0\rangle\langle \tilde{f}_0 | + \int_{C_d} dz |f_z\rangle z \langle \tilde{f}_z |, \quad (37)$$

From this spectral expansion the time-evolution of any vector  $|\psi\rangle \in \Psi$  can be computed as:

$$|\psi_t\rangle = e^{-iHt}|\psi\rangle = e^{-iz_0t} |f_0\rangle\langle \tilde{f}_0 | \psi\rangle + \int_{C_d} dz |f_z\rangle e^{-izt} \langle \tilde{f}_z | \psi\rangle, \quad (38)$$

where we can identify an exponentially decaying component  $|f_0\rangle$ , that we would like to associate to an unstable state or Gamov vector.

We can also choose a curve  $\Gamma_u$  as in fig. 3<sub>a</sub>, and in this case we obtain the contribution of the two curves  $C_u$  and  $C'_u$  as shown in fig. 3<sub>b</sub>. The new spectral decomposition so obtained reads:

$$\begin{aligned} I &= \int_{\Gamma_u} dz |z_+\rangle \langle z_+| = \int_{C_u} dz |z_+\rangle \langle z_+| + \int_{C'_u} dz |z_+\rangle \langle z_-| S^\#(z) = \\ &= \int_{C_u} dz |\tilde{f}_z\rangle \langle f_z| + |\tilde{f}_0\rangle \langle f_0| \end{aligned} \quad (39)$$

where:

$$\begin{aligned} |\tilde{f}_z\rangle &= |z_+\rangle, & \langle f_z| &= \langle z_+| \\ |\tilde{f}_0\rangle &= [2\pi i (\text{Res} S^\#) z_0^*]^{\frac{1}{2}} |z_{0+}^*\rangle, & \langle f_0| &= [2\pi i (\text{Res} S^\#) z_0^*]^{\frac{1}{2}} \langle z_{0-}^*|, \end{aligned} \quad (40)$$

Using eqs. 2.33 and 2.23 we can also obtain:

$$\begin{aligned} \langle f_z | \tilde{f}_{z'} \rangle &= \delta_{C_u}(z - z'), & \langle f_0 | \tilde{f}_0 \rangle &= 1, \\ \langle f_0 | \tilde{f}_{z'} \rangle &= \langle f_z | \tilde{f}_0 \rangle = 0. \end{aligned} \quad (41)$$

In this representation the spectral expansion of the Hamiltonian is:

$$H = z_0^* |\tilde{f}_0\rangle \langle f_0| + \int_{C_u} dz |\tilde{f}_z\rangle z \langle f_z|, \quad (42)$$

and the evolution of a vector  $|\varphi\rangle \in \Phi$  is given by:

$$|\varphi_t\rangle = e^{-iHt} |\varphi\rangle = e^{-iz_0^* t} |\tilde{f}_0\rangle \langle f_0| \varphi\rangle + \int_{C_u} dz e^{-izt} |\tilde{f}_z\rangle \langle f_z| \varphi\rangle, \quad (43)$$

In this case we find a growing component  $|\tilde{f}_0\rangle$  that we shall identify also with an unstable state.

In this section  $\Psi$  is the space of vectors  $\psi$  for which  $\langle z_+ | \psi \rangle$  is analytic in the region between  $C_d$  and  $\mathcal{R}^+$  of figure 2b, satisfying  $e^{-iHt} \Psi \subset \Psi$ , since if  $\langle z_+ | \psi \rangle \in \Psi$   $\langle z_+ | e^{-iHt} \psi \rangle = e^{-izt} \langle z_+ | \psi \rangle$  and therefore the time evolution given by 2.32 is valid for all values of  $t \in \mathcal{R}$ .

Also  $\Phi$  is the set of vectors  $\varphi$  for which  $\langle z_+ | \varphi \rangle$  is analytic in the region between  $C_u$  and  $\mathcal{R}^+$  of figure 3b,  $e^{-iHt} \Phi \subset \Phi$  and the time evolution given by 2.37 is valid for all values of  $t \in \mathcal{R}$ .

If further restrictions are imposed on the vector spaces  $\Psi$  and  $\Phi$ , the time evolutions 2.32 and 2.37 will be valid only for restricted values of  $t$ . This possibility is discussed in sections 2.5 and 4.

The formal developments of this section are applied in section 5 to Friedrichs model.

## 2.5 Generalized expansions in the literature.

The generalized spectral decompositions 2.31 and 2.36, obtained above, appear in the literature originated by different approaches:

-Sudarshan et al. [4] proposed a generalized quantum formulation using analytic continuations defined, from the beginning, on a curve in the complex plane, like  $\Gamma$ , instead of the real semiaxis.

-A. Bohm et al. [5] considered a formulation of quantum mechanics in rigged Hilbert spaces and fix  $\Gamma = (-\infty, 0]$ . In this approach expressions 2.31 and 2.32 correspond to vectors  $|\psi\rangle \in \phi_- \subset \Psi \subset \mathcal{S} \subset \mathcal{H}$ , being  $\phi_-$  the set of vectors  $|\psi\rangle$  such that  $\langle\omega_+|\psi\rangle$  is a function belonging to the Hardy class from below (cf. section 4). Then if  $|\psi\rangle \in \phi_-$ ,  $e^{-iHt}|\psi\rangle \in \phi_-$  only for  $t > 0$ . Expressions 2.36 and 2.37 apply to vectors  $|\varphi\rangle \in \phi_+ \subset \Phi \subset \mathcal{S} \subset \mathcal{H}$ , where  $\phi_+$  is the set of vectors  $|\varphi\rangle$  such that the function  $\langle\omega_+|\varphi\rangle$  belongs to the Hardy class from above. Then if  $|\varphi\rangle \in \phi_+$ ,  $e^{-iHt}|\varphi\rangle \in \phi_+$  only for  $t < 0$ . Thus the time evolution is decomposed in two semigroups, being this fact the main advantage of Bhom proposal.

-The same representations are obtained by Petrosky et al. [8], for the Friedrichs model, using a perturbative scheme together with a time ordering rule. The interpretation of this approach in terms of rigged Hilbert spaces is given by Antoniou et al. [6].

-More recently, A. Bohm et al. [9] and A. Bohm [10] deduced the need of Hardy class functions from a 'quantum arrow of time', stating that measurements can only be realized after preparation of states.

-A more general mathematical structure (doublets) to represent Gamov vectors is proposed in reference [11]

The results presented in the previous subsection can be compared with the ones of references [6] and [8] if we define:

$$\begin{aligned} |f_\omega\rangle &= S(\omega) |\omega_-\rangle, & \langle \tilde{f}_\omega | &= \langle \omega_+ |, \\ | \tilde{f}_\omega \rangle &= |\omega_+\rangle, & \langle f_\omega | &= \langle \omega_- | S^*(\omega) \end{aligned} \tag{44}$$

where  $S(\omega)$  and  $S^*(\omega)$  are the distributions defined by:

$$\int_0^\infty d\omega S(\omega) \psi(\omega) = \int_0^\infty d\omega S(\omega) \psi(\omega) + 2\pi i (\text{Res} S)_{z_0} \psi(z_0),$$

$$\int_0^\infty d\omega S^*(\omega) \varphi(\omega) = \int_0^\infty d\omega S^*(\omega) \varphi(\omega) - 2\pi i (Res S^\#)_{z_0^*} \varphi(z_0^*), \quad (45)$$

where  $\psi(\omega) \in \Psi$  and  $\varphi(\omega) \in \Phi$ .

Using eqs. 2.38 and 2.39, we can rewrite eqs.2.31, 2.32, 2.36, and 2.37 as:

$$H = z_0 |f_0\rangle \langle \tilde{f}_0| + \int_0^\infty d\omega |f_\omega\rangle \omega \langle \tilde{f}_\omega|, \quad (46)$$

$$e^{-iHt} |\psi\rangle = e^{-iz_0 t} |f_0\rangle \langle \tilde{f}_0| \psi\rangle + \int_0^\infty d\omega e^{-i\omega t} |f_\omega\rangle \langle \tilde{f}_\omega| \psi\rangle, \quad (47)$$

$$H = z_0^* | \tilde{f}_0\rangle \langle f_0| + \int_0^\infty d\omega | \tilde{f}_\omega\rangle \omega \langle f_\omega|, \quad (48)$$

$$e^{-iHt} |\varphi\rangle = e^{-iz_0^* t} | \tilde{f}_0\rangle \langle f_0| \varphi\rangle + \int_0^\infty d\omega e^{-i\omega t} | \tilde{f}_\omega\rangle \langle f_\omega| \varphi\rangle. \quad (49)$$

Orthogonality conditions are:

$$\begin{aligned} \langle \tilde{f}_0 | f_0 \rangle &= 1, & \langle \tilde{f}_\omega | f_{\omega'} \rangle &= \delta(\omega - \omega'), \\ \langle \tilde{f}_0 | f_\omega \rangle &= \langle \tilde{f}_\omega | f_0 \rangle = 0, \\ \langle f_0 | \tilde{f}_0 \rangle &= 1, & \langle f_\omega | \tilde{f}_{\omega'} \rangle &= \delta(\omega - \omega'), \\ \langle f_0 | \tilde{f}_\omega \rangle &= \langle f_\omega | \tilde{f}_0 \rangle = 0. \end{aligned} \quad (50)$$

### 3 The norm of Gamov vectors.

Let us consider again eq. 2.32 and the decaying component  $|f_0\rangle$ . The exponential decay is usually obtained in quantum mechanics as an approximation given by the Fermi golden rule. However, for very short and very large times, quantum mechanics predicts a deviation from exponential behavior. As the life-time of some unstable states can be very large, and the exponential decay is measured with high precision, there has been strong interest to consider generalized spectral decompositions of the Hamiltonian, with complex eigenvalues, so that the corresponding eigenvectors could describe unstable states with exact exponential decay, namely the component  $|f_0\rangle$  that we have found in eq 2.32. In order to precise the nature of this kind of states it is interesting to compute their norm and their mean energy.

Let us consider the "Gamov vector"  $|f_0\rangle$  given by eq. 2.28, and let us try to compute the norm and the energy of this state. The vectors  $|f_0\rangle$  and  $\langle f_0|$  defined in eqs. 2.28 and 2.34 can be written as:

$$\begin{aligned} |f_0\rangle &= [-2\pi i(Res S)_{z_0}]^{-\frac{1}{2}} \oint_{C'_d} |z_+\rangle dz, \\ \langle f_0| &= [2\pi i(Res S^\#)_{z_0^*}]^{-\frac{1}{2}} \oint_{C'_u} \langle z_+| dz. \end{aligned} \quad (51)$$

The integrals over  $C'_d$  and  $C'_u$  can be deformed into a single closed curve  $C$ , as shown in fig. 4<sup>2</sup>. Then using eqs 51 and 2.23 we obtain:

$$\begin{aligned} \langle f_0|f_0\rangle &\sim \oint_{-C} dz \langle z_+| \oint_C dz' |z'_+\rangle = \\ &= - \oint_C dz \oint_C dz' \delta_C(z - z') = - \oint_C dz = 0. \end{aligned} \quad (52)$$

The same result can be obtained using the orthogonality condition 2.44:

$$\begin{aligned} \langle f_0|f_0\rangle &= \langle f_0|I|f_0\rangle = \langle f_0| \int_0^\infty d\omega |\omega_+\rangle \langle \omega_+| f_0\rangle = \\ &= \int_0^\infty d\omega \langle f_0| \tilde{f}_\omega \rangle \langle \tilde{f}_\omega | f_0\rangle = 0, \end{aligned} \quad (53)$$

and also:

$$\langle f_0|H|f_0\rangle = \int_0^\infty d\omega \omega \langle f_0| \tilde{f}_\omega \rangle \langle \tilde{f}_\omega | f_0\rangle = 0, \quad (54)$$

We shall check these results again in section 5 in the case of the Friedrichs model.

The fact that the generalized state  $|f_0\rangle$  have zero norm and zero energy seems to indicate that it is not a physical state, in fact:

i.-It is defined just as a functional and therefore it belongs to  $\Psi^\times$ , a space of generalized states.

ii.- $|f_0\rangle$  never appears alone, but only as a component of a regular physical state, as in eqs. 2.32 or 2.41. Thus  $|f_0\rangle$  could be considered as a "ghost".

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<sup>2</sup>This deformation can be done provided spaces  $\Phi$  and  $\Psi$  are chosen in such a way that the curve  $C$  is contained in the domain where the test functions  $\langle \varphi|z\rangle$  ( $\langle z|\psi\rangle$ ) of  $|f_0\rangle$  ( $\langle f_0|$ ) are analytic. It is also necessary that  $|z_+\rangle$  and  $\langle z_+|$  remain analytic during the deformation. We shall check these requirements for Friedrichs model in section 5.

Like Fadeev-Popov ghosts, it is only useful to perform certain calculation, precisely the generalized spectral decompositions, in our case.

The functional  $\langle \widetilde{f}_0 |$  has the same properties.

Moreover these results are essential for the internal coherence of the theory for two very important reasons:

i.-Eqs. 3.2 and 3.4 are necessary conditions for the conservation of probability and energy within the theory. In fact the terms like  $e^{-iHt}|f_0\rangle = e^{-iz_0t}|f_0\rangle$  vanish when  $t \rightarrow \infty$ , therefore they must have vanishing norm and energy, since these quantities must be constant in time.

ii.-For mixed states, eqs. 3.2 and 3.4 appear again. Traces and mean values of "fluctuations" vanish, and this fact is also essential for the consistency of the formalism, as we shall show in [7].

## 4 Time evolution and time asymmetry.

A. Bohm et al., in reference [5], [9] and [10] proposed a special choice for the spaces we have called  $\Phi$  and  $\Psi$ . Precisely they proposed that these spaces must coincide with spaces  $\phi_+$  and  $\phi_-$  defined as the sets of vectors with analytic extension to the upper or the lower half plane, or in mathematical terms:

$$\phi_{\pm} = \{|\psi\rangle / \langle \omega_{\mp} | \psi \rangle \in \theta(\mathcal{S} \cap H_{\pm}^2)\}, \quad (55)$$

where  $\mathcal{S}$  denotes the Schwarz class,  $H_{\pm}^2$  the upper (lower) Hardy class, and  $\theta$  is the Heaviside step function.

A complex function  $f(\omega)$  on  $\mathcal{R}$  is a Hardy class function from above (below) if:

i.- $f(\omega)$  is the boundary value of a function  $f(z)$  of complex variable  $z = x + iy$  that is analytic in the half plane  $y > 0$  ( $y < 0$ ).

ii.- $\int_{-\infty}^{+\infty} |f(x + iy)|^2 dx < k < \infty$ , for all  $y$  such that  $0 < y < \infty$ , ( $-\infty < y < 0$ ).

Then,  $|\psi\rangle \in \phi_- \Rightarrow e^{-iHt}|\psi\rangle \in \phi_-$  if  $t > 0$ , ([5],[6]) and the time evolution of  $|\psi\rangle$  can be computed using eqs. 2.32 or 2.41 where  $|f_0\rangle$ ,  $\langle \widetilde{f}_0 |$ ,  $|f_{\omega}\rangle$ , and  $\langle \widetilde{f}_{\omega} |$  are generalized states of the dual space  $\phi_-^{\times}$ . Thus:

$$e^{-iHt}\phi_- \subset \phi_-, \quad \text{if } t > 0, \quad (56)$$

Also  $|\varphi\rangle \in \phi_+ \Rightarrow e^{-iHt}|\varphi\rangle \in \phi_+$  if  $t < 0$ , ([5],[6]) and the time evolution of  $|\varphi\rangle$  can be computed using eqs. 2.37 or 2.43, being  $|\tilde{f}_0\rangle$ ,  $\langle f_0|$ ,  $|\tilde{f}_\omega\rangle$ , and  $\langle f_\omega|$  generalized states of the dual space  $\phi_+^\times$ . Thus:

$$e^{-iHt}\phi_+ \subset \phi_+, \quad \text{if } t < 0. \quad (57)$$

In the first case the Gamov vector  $|f_0\rangle$  decays toward the future, while in the second case the Gamov vector  $|\tilde{f}_0\rangle$  decays towards the past.

The demonstration of this properties, given in reference [5], is based in two theorems:

**-Paley-Wiener theorem.**

If  $\varphi(\omega) \in H_-^2$  (the Hardy class from below) then the Fourier transform:

$$[\mathcal{F}\varphi(\omega)]_s = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp(-is\omega)\varphi(\omega)d\omega$$

is endowed with the property:

$$[\mathcal{F}\varphi(\omega)]_s = 0, \quad \text{if } s > 0.$$

**-Theorem:**

If  $\varphi(\omega) \in H_-^2$  then  $\exp(-i\omega t)\varphi(\omega) \in H_-^2$  if  $t > 0$ .

In fact, if  $\varphi(\omega) \in H_-^2$  then:

$$[\mathcal{F}\exp(-i\omega t)\varphi(\omega)]_s = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp[-i\omega(s+t)]\varphi(\omega)d\omega$$

which is zero if  $s > -t$  and consequently  $\exp(-i\omega t)\varphi(\omega) \in H_-^2$  if  $t > 0$ . Q.E.D.

As both  $\phi_+$  and  $\phi_-$  are dense in  $\mathcal{H}$ , it seems reasonable to represent any physical state as a vector of  $\phi_+$  or  $\phi_-$ , so we will restrict all our physical reasonings to one of these spaces only. In fact, it is quite useless to speculate about the physical nature of spaces  $\phi_+^\times$  or  $\phi_-^\times$ , since all the physic is really contained, as we shall see, in one of the two test functions spaces  $\phi_+$  or  $\phi_-$  and furthermore these spaces are dense in  $\phi_+^\times$  and  $\phi_-^\times$ .

Let us now analyze the action of the time inversion operator  $K$  in the spaces  $\phi_+$  or  $\phi_-$ . For any  $|\psi\rangle$  represented in terms of the  $|\omega\rangle$  ( $H_0|\omega\rangle = \omega|\omega\rangle$ ) the time-inversion operator  $K$  is defined by:

$$K|\psi\rangle = \int_0^\infty d\omega |\omega\rangle \langle \omega|\psi\rangle^*, \quad (58)$$

from which we can deduce that  $K|\omega\rangle = |\omega\rangle$ .

The operator  $K$  satisfies:

$$\begin{aligned} K(a_1|\psi_1\rangle + a_2|\psi_2\rangle) &= a_1^*K|\psi_1\rangle + a_2^*K|\psi_2\rangle \\ \langle\varphi|K\psi\rangle &= \langle K\varphi|\psi\rangle^*, \quad K^2 = 1. \end{aligned}$$

If, as usual,  $[H, K] = 0$  (see eq. 1.2) and  $|\psi(t)\rangle$  is a solution of Schroedinger equation,  $K|\psi(-t)\rangle$  is also a solution of this equation, and the quantum model turns out to be time symmetric if we work in  $\mathcal{H}$  space, and we have that:

$$K : \mathcal{H} \rightarrow \mathcal{H}, \quad (59)$$

namely  $K$  maps the space of what in a non rigorous theory are considered physical solutions over the same space (and the same thing happens with  $\mathcal{S}$  in the rigorous theory, cf. eq 4). As we shall see in a moment, this is not the case if the space of the physical states is  $\phi_-$  or  $\phi_+$ .

Assuming  $[H_0, K] = [V, K] = 0$ , together with 58 and the definition of  $|\omega_{\mp}\rangle$  given in 2.20, we obtain:

$$K|\omega_{\pm}\rangle = |\omega_{\mp}\rangle$$

Let  $|\psi\rangle \in \phi_-$ , then  $\langle\omega_+|\psi\rangle \in \theta(\mathcal{S} \cap H_-^2)$ , and

$$K|\psi\rangle = K \int_0^\infty d\omega |\omega_+\rangle \langle\omega_+|\psi\rangle = \int_0^\infty d\omega |\omega_-\rangle \langle\omega_+|\psi\rangle^*$$

from which  $\langle\omega_-|K\psi\rangle = \langle\omega_+|\psi\rangle^* \in \theta(\mathcal{S} \cap H_+^2)$ , and therefore  $K|\psi\rangle \in \phi_+$ . In general we have proved that:

$$K : \phi_{\pm} \rightarrow \phi_{\mp}. \quad (60)$$

We have now all the elements to show that if we postulate that: e. g.  $\phi_-$  is the space of the physical states our theory becomes time-asymmetric, as was stated in the introduction. In fact:

i.-From eq. 60 we see that  $K : \phi_- \rightarrow \phi_+$  so time inversion do not exists within the space of physical states

ii.-If  $t > 0$ , eq. 56 shows that the evolution operator  $U(t) = e^{-iHt}$  exists in the physical space  $\phi_-$ . But eq. 57 shows also that the inverted time operator  $U(t)^{-1} = e^{iHt}$  do not exists in this space of physical state  $\phi_-$ . This is, of course, the essence of an irreversible theory.



Then, with a minimal change we have obtain an irreversible quantum theory and, as in the physical real world it is impossible to invert the time evolution, we can claim that  $\phi_-$  mimic, better than  $\mathcal{H}$ , the physical reality.

We close this section with three observations:

i.-Our model can be a local model, as a decaying (or growing) process, or a global model, as a cosmological model of the universe.

-If it is a local model we will always deal with a period of time  $-\infty < t \leq 0$  when the system is prepared and  $\phi_+$  is chosen as the physical space, and a period  $0 \leq t < \infty$  when the measurement is performed and the physical space is  $\phi_-$  [9] and [10]. Of course in this case we cannot consider the problem of the arrow of time in full, since we can change the roles of  $\phi_+$  and  $\phi_-$  and also we cannot consider e.g. the cosmological arrow of time. Being our model just local, in this case, the real arrow of time is essentially imposed from the exterior of the model, i. e. the rest of the universe.

-In the second case we will consider only the period  $0 \leq t < \infty$  and  $t = 0$  will usually corresponds to the "big bang" time, and we postulate that the universe do not exists before that time. In this case the model is complete but, of course, much more complicated, and the research of this kind of problems is just beginning ([12]). The space of physical states is usually  $\phi_-$  (even if we can also use  $\phi_+$  and the period  $-\infty < t \leq 0$ , as we shall explain in ii). Therefore irreversibility is introduced in the model as we have explained and the problem of the arrows of time in the universe can be explained in full. Once the global arrow of time is established in the universe it can be used to define the local arrows in the subsystems of the universe (using e. g. the Reichenbach branch system [13]).

ii.-Someone may say that we have introduced in the global case the arrow of time "just by hand", when we chose space  $\phi_-$  or  $\phi_+$  as the space of physical states. In order to answer this criticism we must define two important words "conventional" and "substantial". Precisely:

-In mathematics we use to work with identical objects, like points, the two directions of an axis, the two semicones of a light cone etc.

-In physics there are also identical objects, like identical particles, the two spin directions etc.

-When ([14],[15]) we are forced to call by different names two identical objects we will say that we are establishing a *conventional difference*, while

- if we call by different names two different objects we will say that we are establishing a *substantial difference*.

The problem of time asymmetry is that, in all time-symmetric physical theories, usually the difference between past and future is just conventional. In fact, we can change the word "past" by the word "future", in these theories, and nothing changes. But we have the clear psychological filling that the past is substantially different than the future. Thus the problem of the arrow of time is to find theories where past is substantially different than future and such that usual well established physics remains valid. Our minimal irreversible quantum mechanics is one of these theories.

In fact: the difference between  $\phi_-$  and  $\phi_+$  in the global case is just conventional since these two spaces are identical. Thus physics is the same in  $\phi_-$  than in  $\phi_+$ . Think in a cosmological model, life will be the same in the universe of  $\phi_-$  than in the universe of  $\phi_+$ . In fact, since in both models of universe (if completely computed) all the arrows of time must point in the same direction, there is no physical way to decide if we are in one model or the other. Thus the choice between  $\phi_-$  and  $\phi_+$  is just conventional and physically irrelevant. But once this choice is made a substantial difference is established in the model of the universe e. g. the only time evolution operator is  $U(t) = e^{-iHt}$ ,  $t > 0$ , and cannot be inverted, we have equilibrium only towards the future, etc.

Thus the choice between  $\phi_-$  and  $\phi_+$  is trivial and unimportant in the global case, that is why the arrow of time is not introduced by hand. The important choice is between  $\mathcal{H}$  (or  $\mathcal{S}$ ) and  $\phi_-$  (or  $\phi_+$ ) as the space of our physical states. And we are free to make this choice, since a good physical theory begins by the choice of the best mathematical structure to mimic real nature.

iii.- Eq. 52 is valid if the curve  $C$  is contained in a domain where the functions of both spaces  $\Psi$  and  $\Phi$  are analytic. But if  $\Psi = \phi_-$  and  $\Phi = \phi_+$ , since the domain where  $\phi_-$  is analytic do not overlap with the one of  $\phi_+$ , it is impossible to draw the curve  $C$ . Nevertheless if we define the domain of analyticity of the functions of  $\Psi$  and  $\Phi$  in such a way that they overlap and contains the curve  $C$  and if the domain of  $\Psi$  ( $\Phi$ ) contains the lower (upper) half plane, then:

$$\Psi \subset \phi_- \subset \mathcal{H} \subset \phi_-^\times \subset \Psi^\times$$

$$\Phi \subset \phi_+ \subset \mathcal{H} \subset \phi_+^\times \subset \Phi^\times$$

Therefore eq. 52 is valid also if we only consider that  $|f_0\rangle \in \phi_-^\times$  since this space is contained in  $\Psi^\times$  where eq. 52 was demonstrated. Analogously we

can prove that  $|\widetilde{f_0}\rangle$  has also vanishing norm.

## 5 Friedrichs model.

In section 3 we have shown that our theory is feasible, in the sense that the norm and the probability is conserved, and in section 4 we have demonstrated that we are dealing with a quantum irreversible theory. It is now necessary to prove that the physical results, obtained by this theory, are correct. This task have been partially done already and can be found in the literature [4],[5],[6], where it is demonstrated that these results coincide, in general, with the ones of the usual "coarse-graining" theory ([2],[3]). In fact, the main contribution of this paper is only to give a better theoretical foundation to these results. So we close this paper with just one example, the well known Friedrichs model, namely the simplest model of an unstable state coupled with a continuous "radiation" field. We shall show how our theory leads to the decaying of the unstable state of this model and how, what is left of it, reappears in the continuous spectrum, we would say as radiation.

Let us consider the Hamiltonian:

$$H = m|1\rangle\langle 1| + \int_0^\infty d\omega |\omega\rangle\omega\langle\omega| + \int_0^\infty d\omega V_\omega(|\omega\rangle\langle 1| + |1\rangle\langle\omega|), \quad (61)$$

where  $|1\rangle$  is the unstable state,  $\{|\omega\rangle\}$  ( $0 \leq \omega < \infty$ ) can be consider as the states of a continuous set of oscillators that symbolizes a radiation field, and the last term of the r. h. s. of eq. 61 is the interaction term.

For this model Lipmann-Schwinger equations 2.20 can be solved exactly giving:

$$\begin{aligned} |\omega_\pm\rangle &= |\omega\rangle + \frac{V_\omega}{\eta(\omega \pm i0)} \left[ |1\rangle - \int_0^\infty \frac{d\omega' V_{\omega'} |\omega'\rangle}{\omega' - (\omega \pm i0)} \right], \\ \langle\omega_\pm| &= \langle\omega| + \frac{V_\omega}{\eta(\omega \mp i0)} \left[ \langle 1| - \int_0^\infty \frac{d\omega' V_{\omega'} \langle\omega'|}{\omega' - (\omega \mp i0)} \right], \end{aligned} \quad (62)$$

where:

$$\eta(z) = z - m + \int_0^\infty \frac{d\omega' V_{\omega'}^2}{\omega' - z}, \quad z \in \mathcal{C} - \mathcal{R}^+ \quad (63)$$

If we consider the set of vectors  $\psi \in \mathcal{C} \times \mathcal{S}$  such that  $\langle 1|\psi\rangle \in \mathcal{C}$  and  $\langle\omega|\psi\rangle \in \mathcal{S}$ , the vectors  $|\omega_\pm\rangle$  ( $\langle\omega_\pm|$ ) are antilinear (linear) functionals acting

on  $\mathcal{C} \times \mathcal{S}$ , and generalized right (left) eigenvectors of the Hamiltonian 5.1:

$$H|\omega_{\pm}\rangle = \omega|\omega_{\pm}\rangle, \quad \langle\omega_{\pm}|H = \omega\langle\omega_{\pm}| \quad (64)$$

The vectors  $|\omega_{+}\rangle$  and  $|\omega_{-}\rangle$  are related by

$$|\omega_{+}\rangle = S(\omega)|\omega_{-}\rangle, \quad S(\omega) = \frac{\eta(\omega - i0)}{\eta(\omega + i0)}$$

If  $\eta(z)$  do not vanish for real values of  $z$ , it is possible to prove [4] that they form a complete biorthogonal set, in the sense that for any two vectors  $\varphi$  and  $\psi$  in  $\mathcal{C} \times \mathcal{S}$  results

$$\langle\varphi|\psi\rangle = \langle\varphi|1\rangle\langle 1|\psi\rangle + \int d\omega \langle\varphi|\omega\rangle\langle\omega|\psi\rangle = \int d\omega \langle\varphi|\omega_{\pm}\rangle\langle\omega_{\pm}|\psi\rangle,$$

and therefore

$$I = \int d\omega |\omega_{\pm}\rangle\langle\omega_{\pm}|, \quad \langle\omega_{\pm}|\omega'_{\pm}\rangle = \delta(\omega - \omega').$$

As  $\eta(z)$  defined by 5.3 has a cut in  $\mathcal{R}^{+}$ , it is possible to define the extension  $\eta_{+}(z)$  ( $\eta_{-}(z)$ ) from the upper to the lower (lower to the upper) half plane as

$$\begin{aligned} \eta_{+}(z) &= \begin{cases} \eta(z) & \text{if } \text{Im } z > 0 \\ \eta(z) + 2\pi i V_z^2 & \text{if } \text{Im } z < 0 \end{cases} \\ \eta_{-}(z) &= \begin{cases} \eta(z) - 2\pi i V_z^2 & \text{if } \text{Im } z > 0 \\ \eta(z) & \text{if } \text{Im } z < 0 \end{cases} \end{aligned} \quad (65)$$

We assume that  $\eta_{+}(z) = 0$  has a single solution  $z_0 \in \mathcal{C}^{-}$ , and therefore  $\eta_{-}(z_0^*) = 0$ ,  $z_0^* \in \mathcal{C}^{+}$ . This means that the analytic extension  $S(z) = \frac{\eta_{-}(z)}{\eta_{+}(z)}$  of  $S(\omega) = \frac{\eta(\omega - i0)}{\eta(\omega + i0)}$  is analytic in  $\mathcal{C} - z_0$  with a simple pole at  $z_0 \in \mathcal{C}^{-}$ .

The functionals 5.2 can be analytically extended to the complex plane:

$$|z^{+}\rangle = |z\rangle + \frac{V_z}{\eta_{+}(z)} \left[ |1\rangle - \int_0^{\infty} d\omega' V_{\omega'} |\omega'\rangle \left( \frac{1}{\omega' - s} \right)_z^{+} \right] \quad (66)$$

$$\langle z^{+}| = \langle z| + \frac{V_z}{\eta_{-}(z)} \left[ \langle 1| - \int_0^{\infty} d\omega' V_{\omega'} \langle \omega'| \left( \frac{1}{\omega' - s} \right)_z^{-} \right] \quad (67)$$

In the last expressions we introduced the functionals  $\left(\frac{1}{\omega'-s}\right)_z^+$  and  $\left(\frac{1}{\omega'-s}\right)_z^-$  defined by:

$$\begin{aligned} \int_0^\infty d\omega' \left(\frac{1}{\omega'-s}\right)_z^+ \varphi(\omega') &= \begin{cases} \int_0^\infty d\omega' \frac{1}{\omega'-z} \varphi(\omega') & \text{if } \text{Im} z > 0 \\ \int_0^\infty d\omega' \frac{1}{\omega'-z} \varphi(\omega') + 2\pi i \varphi(z) & \text{if } \text{Im} z < 0 \end{cases} \\ \int_0^\infty d\omega' \left(\frac{1}{\omega'-s}\right)_z^- \psi(\omega') &= \begin{cases} \int_0^\infty d\omega' \frac{1}{\omega'-z} \psi(\omega') - 2\pi i \psi(z) & \text{if } \text{Im} z > 0 \\ \int_0^\infty d\omega' \frac{1}{\omega'-z} \psi(\omega') & \text{if } \text{Im} z < 0 \end{cases} \end{aligned} \quad (68)$$

which are well defined on functions  $\varphi(\omega')$  ( $\psi(\omega')$ ),  $\omega' \in \mathcal{R}^+$ , having analytic extensions to the lower (upper) half plane.

Following the procedure given in section 2, the spectral decompositions 2.31 and 2.36 can be found and the following generalized eigenvectors are obtained ([6],[8]):

$$\begin{aligned} |\tilde{f}_0\rangle &= \frac{1}{\sqrt{\eta'_-(z_0^*)}} \left[ |1\rangle - \int_0^\infty d\omega V_\omega \left(\frac{1}{\omega-s}\right)_{z_0^*}^- |\omega\rangle \right] \\ \langle f_0| &= \frac{1}{\sqrt{\eta'_-(z_0^*)}} \left[ \langle 1| - \int_0^\infty d\omega V_\omega \left(\frac{1}{\omega-s}\right)_{z_0^*}^- \langle \omega| \right] \\ |\tilde{f}_\omega\rangle &= |\omega\rangle + \frac{V_\omega}{\eta_+(\omega)} \left[ |1\rangle + \int_0^\infty \frac{d\omega' V_{\omega'}}{\omega - \omega' + i0} |\omega'\rangle \right] \\ \langle f_\omega| &= \langle \omega| + \frac{V_\omega}{\eta_-(\omega)} \left[ \langle 1| + \int_0^\infty \frac{d\omega' V_{\omega'}}{\omega - \omega' - i0} \langle \omega'| \right], \end{aligned} \quad (69)$$

$$\begin{aligned} |f_0\rangle &= \frac{1}{\sqrt{\eta'_+(z_0)}} \left[ |1\rangle - \int_0^\infty d\omega V_\omega \left(\frac{1}{\omega-s}\right)_{z_0}^+ |\omega\rangle \right] \\ \langle \tilde{f}_0| &= \frac{1}{\sqrt{\eta'_+(z_0)}} \left[ \langle 1| - \int_0^\infty d\omega V_\omega \left(\frac{1}{\omega-s}\right)_{z_0}^+ \langle \omega| \right] \\ |f_\omega\rangle &= |\omega\rangle + \frac{V_\omega}{\eta_+(\omega)} \left[ |1\rangle + \int_0^\infty \frac{d\omega' V_{\omega'}}{\omega - \omega' + i0} |\omega'\rangle \right] \\ \langle \tilde{f}_\omega| &= \langle \omega| + \frac{V_\omega}{\eta_-(\omega)} \left[ \langle 1| + \int_0^\infty \frac{d\omega' V_{\omega'}}{\omega - \omega' - i0} \langle \omega'| \right], \end{aligned} \quad (70)$$

In the previous equations we introduced the distributions  $\frac{1}{\eta_-(\omega)}$  and  $\frac{1}{\eta_+(\omega)}$ , defined by:

$$\begin{aligned}\int_0^\infty d\omega \frac{1}{\eta_+(\omega)} \varphi(\omega) &\equiv \int_0^\infty d\omega \frac{1}{\eta_+(\omega)} \varphi(\omega) + 2\pi i \frac{\varphi(z_0)}{\eta'_+(z_0)}, \\ \int_0^\infty d\omega \frac{1}{\eta_-(\omega)} \psi(\omega) &\equiv \int_0^\infty d\omega \frac{1}{\eta_-(\omega)} \psi(\omega) - 2\pi i \frac{\psi(z_0^*)}{\eta'_-(z_0^*)}.\end{aligned}\quad (71)$$

From the explicit expressions given in 5.9 and 5.10 for  $\langle f_0|$  and  $|f_0\rangle$ , and the orthogonality relations

$$\langle 1|1\rangle = 1, \quad \langle 1|\omega\rangle = \langle \omega|1\rangle = 0, \quad \langle \omega|\omega'\rangle = \delta(\omega - \omega'),$$

we obtain

$$\langle f_0|f_0\rangle = \frac{1}{\sqrt{\eta'_-(z_0^*)\eta'_+(z_0)}} \left[ 1 + \int_0^\infty d\omega V_\omega^2 \left( \frac{1}{\omega - s} \right)_{z_0^*}^- \left( \frac{1}{\omega - s'} \right)_{z_0}^+ \right] \quad (72)$$

Taking into account

$$\frac{1}{\omega - s} \times \frac{1}{\omega - s'} = \frac{1}{s - s'} \left( \frac{1}{\omega - s} - \frac{1}{\omega - s'} \right),$$

and

$$\eta_+(z_0) = z_0 - m + \int_0^\infty d\omega V_\omega^2 \left( \frac{1}{\omega - s'} \right)_{z_0}^+ = 0,$$

$$\eta_-(z_0^*) = z_0^* - m + \int_0^\infty d\omega V_\omega^2 \left( \frac{1}{\omega - s} \right)_{z_0^*}^- = 0,$$

in equation 5.12, we obtain

$$\langle f_0|f_0\rangle = 0.$$

Then, even if it was already proved in section 3, we show again, with this example, that Gamov vector  $|f_0\rangle$  has zero norm.

The spaces  $\phi_+$  ( $\phi_-$ ) in which the analytic extensions to the upper (lower) half plane are well defined in this model are:

$$\phi_\pm = \{|\psi\rangle / \langle 1|\psi\rangle \in \mathcal{C}, \langle \omega|\psi\rangle \in \theta(\mathcal{S} \cap H_\pm^2)\}.$$

It is interesting to note that a state with  $\langle \omega | \psi \rangle = 0$  can be considered to be either in  $\phi_-$  or in  $\phi_+$ , because the analytic extension of this last function is  $\langle z | \psi \rangle = 0$ . Consequently  $|1\rangle \in \phi_- \cap \phi_+$ , and  $e^{-iHt}|1\rangle$  is in  $\phi_-$  for  $t > 0$  and in  $\phi_+$  for  $t < 0$ , therefore it decays to the future as a vector of  $\phi_-$ , and to the past, as a vector of  $\phi_+$ .

Let us now compute the time evolution of the observables  $A$  and their mean values. We have:

$$\langle A \rangle_t = \langle \psi(t) | A | \psi(t) \rangle = \langle \psi | e^{iHt} A e^{-iHt} | \psi \rangle = \langle \psi | A(t) | \psi \rangle. \quad (73)$$

We assume that the observable  $A$  can be written as:

$$\begin{aligned} A = & A_1 |1\rangle \langle 1| + \int_0^\infty d\omega A_\omega |\omega\rangle \langle \omega| + \int_0^\infty d\omega A_{1\omega} |1\rangle \langle \omega| + \\ & + \int_0^\infty d\omega' A_{\omega'1} |\omega'\rangle \langle 1| + \int_0^\infty \int_0^\infty d\omega d\omega' A_{\omega'\omega} |\omega'\rangle \langle \omega|, \end{aligned} \quad (74)$$

where, in the second term, we have included a singular diagonal term, which is present in many observables, like the Hamiltonian of eq. 61. Alternatively we may use the vectors  $|\omega_+\rangle = |\tilde{f}_\omega\rangle$  and  $\langle \omega_+| = \langle \tilde{f}_\omega|$  to represent  $A$  as:

$$A = \int_0^\infty d\omega A_\omega^+ |\tilde{f}_\omega\rangle \langle \tilde{f}_\omega| + \int_0^\infty \int_0^\infty d\omega d\omega' A_{\omega\omega'}^+ |\tilde{f}_\omega\rangle \langle \tilde{f}_{\omega'}| = A_I + A_F. \quad (75)$$

Comparing eq. 75 with eq. 74, and using the explicit expressions for  $|\tilde{f}_\omega\rangle$  and  $\langle \tilde{f}_\omega|$  given by eqs. 5.9 and 5.10 we can prove that  $A_\omega^+ = A_\omega$ , and therefore:

$$\int_0^\infty d\omega \Pi_\omega A = A_I = \int_0^\infty A_\omega |\tilde{f}_\omega\rangle \langle \tilde{f}_\omega|, \quad (76)$$

where:

$$\Pi_\omega A \equiv A_\omega |\tilde{f}_\omega\rangle \langle \tilde{f}_\omega|$$

This is the time invariant part of the observable  $A$  in the Heisemberg picture. Using eqs. 5.15 and 5.4 we obtain:

$$A(t) = e^{iHt} A e^{-iHt} = A_I + e^{iHt} A_F e^{-iHt}. \quad (77)$$

Essentially observables are represented by operators used to compute mean values, as in eq. 5.13. If in this equation  $\psi \in \phi_-$  and we wish to use the spectral decomposition 2.41, it is necessary to assume that the functions

$A_{1\omega}$ ,  $A_{\omega'1}$ , and  $A_{\omega'\omega}$  of eq. 5.14 can be analytically extended to the lower (upper) half plane in the variable  $\omega$  ( $\omega'$ ). (we shall discuss this fact at large in a forthcoming paper [7]). This assumption is compatible with the results of reference [9] and [10]. So we can deduce that:

$$\begin{aligned}
& e^{iHt} A_F e^{-iHt} \\
&= e^{iHt} \left[ |\widetilde{f_0}\rangle\langle f_0| + \int d\omega |\widetilde{f_\omega}\rangle\langle f_\omega| \right] A_F \left[ |f_0\rangle\langle \widetilde{f_0}| + \int d\omega' |f_{\omega'}\rangle\langle \widetilde{f_{\omega'}}| \right] e^{-iHt} \\
&= e^{i(z_0^* - z_0)t} \Pi_{00} A + \int_0^\infty d\omega e^{i(\omega - z_0)t} \Pi_{\omega 0} A + \\
&\quad + \int_0^\infty d\omega' e^{i(z_0^* - \omega')t} \Pi_{0\omega'} A + \int_0^\infty \int_0^\infty e^{i(\omega - \omega')t} \Pi_{\omega\omega'} A,
\end{aligned} \tag{78}$$

where:

$$\begin{aligned}
\Pi_{00} A &\equiv |\widetilde{f_0}\rangle\langle f_0| A_F |f_0\rangle\langle \widetilde{f_0}|, \\
\Pi_{\omega 0} A &\equiv |\widetilde{f_\omega}\rangle\langle f_\omega| A_F |f_0\rangle\langle \widetilde{f_0}|, \\
\Pi_{0\omega} A &\equiv |\widetilde{f_0}\rangle\langle f_0| A_F |f_\omega\rangle\langle \widetilde{f_\omega}|, \\
\Pi_{\omega\omega'} A &\equiv |\widetilde{f_\omega}\rangle\langle f_\omega| A_F |f_{\omega'}\rangle\langle \widetilde{f_{\omega'}}|.
\end{aligned} \tag{79}$$

From eqs. 5.17 and 5.18 we can deduce the time evolution of an observable, and from eq. 5.13 the time evolution of its mean value.

But in order to understand what really is going on it is interesting to obtain an approximated expression for  $A(t)$ , when the interaction function  $V_\omega$  is small, because in this case the unstable state  $|1\rangle$  will have a long life-time and we will be able to obtain a long pure exponential decay. For this purpose it is necessary to obtain the asymptotic form of the projectors of eqs. 5.16 and 5.19 for weak interactions. Taking into account eq. 5.9 and 5.10 we obtain the weak limit:

$$\lim_{V \rightarrow 0} |\widetilde{f_\omega}\rangle\langle \widetilde{f_\omega}| = |\omega\rangle\langle \omega| + \lim_{V \rightarrow 0} \frac{V_\omega^2}{\eta_+(\omega)\eta_-(\omega)} |1\rangle\langle 1|$$

Then, from eq (5.3) we have:

$$\frac{V_\omega^2}{\eta_+(\omega)\eta_-(\omega)} = \frac{V_\omega^2}{(\omega - m - \Delta)(\omega - m - \Delta^*)} = \frac{V_\omega^2}{\Delta^* - \Delta} \left( \frac{1}{\omega - m - \Delta^*} - \frac{1}{\omega - m - \Delta} \right)$$

where:

$$\Delta \equiv \int_0^\infty \frac{d\omega' V_{\omega'}^2}{\omega + i0 - \omega'}.$$



Therefore:

$$\Delta^* - \Delta = \int_0^\infty d\omega' V_{\omega'}^2 \left( \frac{1}{\omega - \omega' - io} - \frac{1}{\omega - \omega' + io} \right) = 2\pi i V_\omega^2$$

As  $Im\Delta^* \rightarrow 0$  and  $\lim_{V \rightarrow 0} \Delta^* = io^+$ , we obtain:

$$\lim_{V \rightarrow 0} \frac{V_\omega^2}{\eta_+(\omega)\eta_-(\omega)} = \frac{1}{2\pi i} \left( \frac{1}{\omega - m - io} - \frac{1}{\omega - m + io} \right) = \delta(\omega - m)$$

and therefore:

$$\lim_{V \rightarrow 0} |\widetilde{f_\omega}\rangle \langle \widetilde{f_\omega}| = |\omega\rangle \langle \omega| + \delta(\omega - m)|1\rangle \langle 1|. \quad (80)$$

From eqs. 5.9 and 5.10 we have:

$$\lim_{V \rightarrow 0} |\widetilde{f_0}\rangle \langle f_0| = \lim_{V \rightarrow 0} |f_0\rangle \langle \widetilde{f_0}| = |1\rangle \langle 1|, \quad (81)$$

$$\lim_{V \rightarrow 0} |\widetilde{f_\omega}\rangle \langle f_\omega| = \lim_{V \rightarrow 0} |f_\omega\rangle \langle \widetilde{f_\omega}| = |\omega\rangle \langle \omega|, \quad (82)$$

These results can be used to obtain the limits of the projectors of eqs. 5.16 and 5.19:

$$\begin{aligned} \lim_{V \rightarrow 0} \Pi_\omega A &= A_\omega[|\omega\rangle \langle \omega| + \delta(\omega - m)|1\rangle \langle 1|] \\ \lim_{V \rightarrow 0} \Pi_{00} A &= (A_1 - A_{\omega=m})|1\rangle \langle 1|, \\ \lim_{V \rightarrow 0} \Pi_{0\omega} A &= A_{1\omega}|1\rangle \langle \omega|, \\ \lim_{V \rightarrow 0} \Pi_{\omega 0} A &= A_{\omega 1}|\omega\rangle \langle 1|, \\ \lim_{V \rightarrow 0} \Pi_{\omega\omega'} A &= A_{\omega\omega'}|\omega\rangle \langle \omega'|. \end{aligned} \quad (83)$$

When the interaction vanishes  $z_0 \rightarrow m$ . However this would be a bad choice for the values in eq. 5.18 if we want to know the approximate behavior of  $A(t)$  for  $t \rightarrow \infty$ . Precisely, if we solve:

$$\eta_+(z_0) = z_0 - m - \int_0^\infty d\omega V_\omega^2 \left( \frac{1}{s - \omega} \right)_{z_0}^+ = 0$$

up to the second order, we obtain:

$$z_0 \cong m + \int_0^\infty \frac{d\omega V_\omega^2}{m + io - \omega}, \quad z_0^* - z_0 \cong 2\pi i V_m^2, \quad (84)$$

Replacing the results of eqs. 5.23 and 5.24 in eqs. 5.18 and 5.16 we obtain the approximate behavior of  $A(t)$  when  $V \rightarrow 0$ , and  $t \rightarrow \infty$ , in such a way that  $V_m^2 t$  is finite. E.g.: if we choose  $A = |1\rangle\langle 1|$  or  $A = |\omega\rangle\langle \omega|$  (or in a more rigorous way  $A = \int f(\omega)|\omega\rangle\langle \omega|d\omega$ ), we can compute the probability to find the system in the unstable states  $|1\rangle$  or in the continuous "radiation" field  $|\omega\rangle$  for large  $t$  and small  $V$ , precisely:

$$\langle \psi(t)|1\rangle\langle 1|\psi(t)\rangle \cong e^{-2\pi V_m^2 t} \langle \psi|1\rangle\langle 1|\psi\rangle$$

$$\langle \psi(t)|\omega\rangle\langle \omega|\psi(t)\rangle \cong \langle \psi|\omega\rangle\langle \omega|\psi\rangle + (1 - e^{-2\pi V_m^2 t})\delta(\omega - m)\langle \psi|1\rangle\langle 1|\psi\rangle. \quad (85)$$

These equations clearly show the exponential decay of the unstable state  $|1\rangle$  and the simultaneous appearance of a radiation state at the energy  $\omega = m$ , namely the radiation outcome of the unstable state. This is a completely reasonable and experimentally verified physical result. However, we must observe that in this case the pure exponential behavior is only a consequence of the approximation we have used.

## 6 Conclusion.

Peter Bergmann said in 1967([16]):

*"It is not very difficult to show that the combination of the reversible laws of mechanics with Gibbsian statistics does not lead to irreversibility but that the notion of irreversibility must be added as a special ingredient..."*

*...the explanation of irreversibility in nature is to my mind still open"*

In fact, from a reversible classical or quantum theory it is impossible to obtain an irreversible one, making only mathematical manipulation. The theory will remain always reversible. So necessarily a *new ingredient* must be added. From 1967 these ingredients were found and classified: coarse-graining, traces, stochastic noises, etc. [16],[17]. The problem is to know what is the minimal ingredient that produces irreversibility in the more aesthetic and economical way. We propose that this minimal ingredient, is to change space  $\mathcal{S}$ , satisfying 4, by space  $\phi_-$ , satisfying 5. This modification can be done in the microscopical quantum level and it is simply to change the space of the regular physical states, as we have done in this paper. The deep physical meaning of this change is the following:

Experimentally we can only perform a finite number of measurements, so that, when we (indirectly) measure a quantum state we only know a

finite number of points (or data) of the corresponding wave function. As we want to have the whole wave function, because, e. g., we cannot find the derivative of a set of finite points, we interpolate this set with a function endowed with mathematical properties that we can freely choose according to our conveniences. We can choose this function in space  $\mathcal{S}$ , namely to use the Gel'fand triplet 1, and then we will obtain the usual reversible quantum mechanics. In this case, if we want to take into account irreversible processes using the usual formalism, we are forced to coarse-grain the system. But we can directly interpolate using functions of the space  $\phi_-$ , namely using the Gel'fand triplet 6, and we will obtain an irreversible quantum mechanics, from the very beginning. Clearly the second process is more economical than the first one.

Thus, the physical basis of the two approaches is the same, we always have only a finite amount of information, but the way to deal with this fact is different. (We will farther discuss these matters elsewhere.) As far as we know the two approaches yield the same physical results, since up to now we do not know of a "cross experiment" to tell us which formalism is the good one. So both theories seem physically equivalent. But, even if the first theory perhaps is more intuitive, the second one has two advantages:

- i.-It contains just one fundamental modification, as we have explained.
- ii.-It provides us with a very simple and powerful computational method: the spectral decomposition in Gamov vectors and the corresponding time evolution (eqs. 2.32, 2.37), obtained using just analytic continuation, which is much more easy to handle than e.g. Feynman path integral, used in the first theory.

So we believe that now the reader knows almost all the features of the problem and he can reach to a final decision by himself.

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## 9 Figure caption.

fig.1: The curve  $\Gamma$ .

fig. 2<sub>a</sub>: The curve  $\Gamma_d$ .

fig. 2<sub>b</sub>. The curves  $C_d$  and  $C'_d$ .

fig. 3<sub>a</sub>. The curve  $\Gamma_u$ .

fig. 3<sub>b</sub>. The curves  $C_u$  and  $C'_u$ .

fig. 4. The curve  $C$ .